CATEGORY THEORY
CATEGORY I - POSETS

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1. Posets

Definition 1 (Objects)**.** A *poset* $(P, \leq)$ consists of a set P together with a relation $\leq$ on P with the following properties:

(P1) $a \leq a$ (Reflexivity)
(P2) $a \leq b$ and $b \leq a$ implies $a = b$ (Antisymmetry)
(P3) $a \leq b$ and $b \leq c$ implies $a \leq c$ (Transitivity)

The relation $\leq$ is called a *partial order* on P, and the word poset is short for "partially ordered set".

A partial order is a *total order* (or *linear order*) if, additionally, the relation satisfies

(P4) $a \leq b$ or $b \leq a$ (Definiteness)

A set together with a total order is call a "totally ordered set".

There are three archetypical examples of posets.

Example 1. Consider the set $\mathbb{R}$ of real numbers, and its standard order relation $\leq$. Then $\leq$ is a total order on $\mathbb{R}$, and $(\mathbb{R}, \leq)$ is a poset. In fact, it is a totally ordered set.

Example 2. Let $a, b \in \mathbb{Z}$. We say that a *divides* b, and write $a \mid b$, if there exists $k \in \mathbb{Z}$ such that $b = ka$.

Let P be a set of positive integers. Then divisibility is a partial order on P, and $(P, |)$ is a poset.

Example 3. Let A and B bet set. We say that A *is contained in* B, and write $A \subset B$, if every element in A is an element in B.

Let $\mathcal{C}$ be a collection of sets. Then containment is a partial order on $\mathcal{C}$. Thus, $(\mathcal{C}, \subset)$ is a poset.

Definition 2 (Subobjects)**.** Let $(P, \leq)$ be a poset. If $Q \subset P$, the restriction of $\leq$ to Q satisfies the properties of a partial order on Q, making $(Q, \leq)$ a poset. We may call $(Q, \leq)$, or just Q, a *subposet*.

Definition 3 (Morphisms)**.** Let P and Q be posets. A function $f : P \to Q$ is called *order preserving* if

$$p_1 \leq p_2 \quad \Rightarrow \quad f(p_1) \leq f(p_2).$$

The identity map on P is order preserving, and the composition of order preserving functions is order preserving. Thus, posets with order preserving maps form a category.

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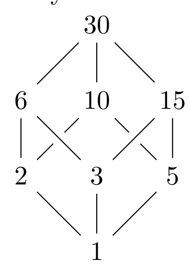
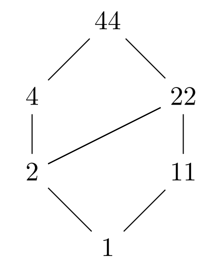
2. Factor Sets

Definition 4 (Objects)**.** Let n be a positive integer. The *factor set* of n, denoted $\mathcal{F}(n)$, is the set of all positive integers which divide n. Factor sets are partially ordered by divisibility.

Finite posets may be drawn using *Hasse diagrams*, as follows.



Factor Set of $n = 30$



Factor Set of $n = 44$

Definition 5 (Morphisms)**.** Let m and n be positive integers. A *morphism* from $\mathcal{F}(n)$ to $\mathcal{F}(m)$ is a function

$$f : \mathcal{F}(n) \to \mathcal{F}(m)$$

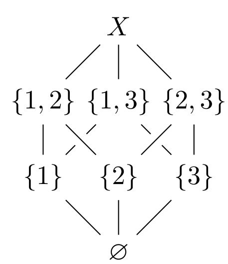
such that,

(a) $f(1) = 1$;
(b) $a \mid b \;\Rightarrow\; f(a) \mid f(b)$, for all $a, b \in \mathcal{F}(n)$.

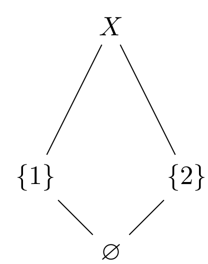
Factor sets form a nonfull subcategory of the category of partially ordered sets.

3. Power Sets

Definition 6 (Objects)**.** Let X be any set. The *power set* of X, denoted $\mathcal{P}(X)$, is the set of all subsets of X. Power sets are partially ordered by containment.



Power set of $X = \{1, 2, 3\}$



Power set of $X = \{1, 2\}$

Definition 7 (Morphisms)**.** Let X and Y be sets. A *morphism* from $\mathcal{P}(X)$ to $\mathcal{P}(Y)$ is a function

$$f : \mathcal{P}(X) \to \mathcal{P}(Y)$$

such that,

(a) $f(\varnothing) = \varnothing$;
(b) $A \subset B \;\Rightarrow\; f(A) \mid f(B)$, for all $A, B \subset X$.

Power sets form a nonfull subcategory of the category of partially ordered sets.

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